Implementation Laguerre Pseudo-Spectral Method for Obtaining the Approximate Solution of Fractional Cable Equation

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Abstract— This paper is devoted to present the approximate solution for the fractional Cable equation (FCE) using an efficient numerical method. The proposed method depends on implementation an approximate formula of the Caputo fractional derivative derived in [14]. This proposed formula is based on the spectral collocation method with the generalized Laguerre polynomials. The properties of these polynomials are used to reduce FCE to solve a system of ODEs which solved using finite difference method. Special attention is given to present the convergence analysis of the given formula. Numerical example is given to show the validity and the accuracy of the proposed algorithm.

Index Terms— Fractional Cable equation, Caputo fractional derivative, Finite difference method, Laguerre polynomials, Laguerre pseudo-spectral method, Convergence analysis.

1 Introduction

Fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering [17]. Consequently, considerable attention has been given to the solutions of FDEs of physical interest. Most FDEs do not have exact solutions, so approximate and numerical techniques ([23], [27]), must be used. Recently, several numerical methods to solve FDEs have been given such as variational iteration method [8],, Adomian decomposition method [5], collocation method ([7], [9]-[13], [22], [26]) and finite difference method ([24], [25]).

The Cable equation is one of the most fundamental equations for modelling neuronal dynamics. Due to its significant deviation from the dynamics of Brownian motion, the anomalous diffusion in biological systems can not be adequately described by the traditional Nernst-Planck equation or its simplification, the Cable equation [15]. Very recently, a modified Cable equation was introduced for modeling the anomalous diffusion in spiny neuronal dendrites [4]. The resulting governing equation, the so-called fractional Cable equation, which is similar to the traditional Cable equation except that the order of derivative with respect to the space and/or time is fractional.

We consider the initial-boundary value problem of the fractional Cable equation which is usually written in the following form

$$u_t(x,t) = D^{\beta}u(x,t) - \mu D^{\gamma}u(x,t) + f(x,t), \quad 0 < x < 1, \quad 0 \le t \le T, \tag{1}$$

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where $0 < \beta \le 2,\ 0 < \gamma \le 1,\ f(x,t)$ is the source term, μ is a constant and D^{δ} is the Caputo fractional derivative with respect to x and of order δ, where $\delta = \beta, \gamma$.

Under the zero boundary conditions

$$u(0,t) = u(1,t) = 0, \tag{2}$$

and the following initial condition

$$u(x,0) = g(x). \tag{3}$$

In the last few years appeared many papers to study this model ([2], [18], [19], [21]), the most of these papers study the ordinary case of such problem but in this paper we study the fractional case.

Our idea is to apply the Laguerre collocation method to discretize (1) to get a linear system of ODEs thus greatly simplifying the problem, and use FDM [20] to solve the resulting system.

2 Preliminaries and Notations

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory that will be required in the present paper.

2.1 The Caputo Fractional Derivative

Definition 1.

The Caputo fractional derivative operator D^{ν} *of order* ν *is defined in the following form*

$$D^{\nu} f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\nu-m+1}}\,dt, \qquad \nu > 0, \quad x > 0,$$

where $m-1 < \nu \le m,\ m \in \mathrm{N},\ \Gamma(.)$ *is the Gamma function.*

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D^{\nu}(\theta_1 f(x) + \theta_2 g(x)) = \theta_1 D^{\nu} f(x) + \theta_2 D^{\nu} g(x),$$

where θ_1 and θ_2 are constants. For the Caputo's derivative we have [17]

$$D^{\nu} C = 0, \quad C \text{ is a constant}, \tag{4}$$

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$$D^{\nu}x^{n}=\begin{cases}0, & \text{for } n\in N_0 \text{ and } n<\lceil \nu \rceil;\\ \dfrac{\Gamma(n+1)}{\Gamma(n-\nu+1)}x^{n-\nu}, & \text{for } n\in N_0 \text{ and } n\geq \lceil \nu \rceil.\end{cases} \tag{5}$$

We use the ceiling function $\lceil \nu \rceil$ to denote the smallest integer greater than or equal to ν, and $N_0=\{0,1,2,...\}$. Recall that for $\nu \in N$ the Caputo differential operator coincides with the usual differential operator of integer order.

For more details on fractional derivatives definitions and its properties see [17].

2.2 The Definition and Properties of the Generalized Laguerre Polynomials

The generalized Laguerre polynomials $\left[L_n^{(\alpha)}(x)\right]_{n=0}^{\infty}, \alpha>-1$ are defined on the unbounded interval $[0,\infty)$ and can be determined with the aid of the following recurrence formula

$$(n+1)L_{n+1}^{(\alpha)}(x)+(x-2n-\alpha-1)L_n^{(\alpha)}(x)+(n+\alpha)L_{n-1}^{(\alpha)}(x)=0, \qquad n=1,2,..., \tag{6}$$

where $L_0^{(\alpha)}(x)=1$, and $L_1^{(\alpha)}(x)=\alpha+1-x.$

The analytic form of these polynomials of degree n is given by

$$L_n^{(\alpha)}(x)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n+\alpha}{n-k}x^k, \tag{7}$$

$L_n^{(\alpha)}(0)=\binom{n+\alpha}{n}$. These polynomials are orthogonal on the interval $[0,\infty)$ with respect to the weight function

$w(x)=\dfrac{1}{\Gamma(1+\alpha)}x^{\alpha}e^{-x}$. The orthogonality relation is

$$\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}x^{\alpha}e^{-x}L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)dx=\binom{n+\alpha}{n}\delta_{nm}. \tag{8}$$

Also, they satisfy the differentiation formula

$$D^kL_n^{(\alpha)}(x)=(-1)^k L_{n-k}^{(\alpha+k)}(x), \quad k=0,1,...,n. \tag{9}$$

Any function $u(x)$ belongs to the space $L_w^2[0,\infty)$ of all square integrable functions on $[0,\infty)$ with weight function $w(x)$, can be expanded in the following Laguerre series

$$u(x)=\sum_{i=0}^{\infty}c_iL_i^{(\alpha)}(x), \tag{10}$$

where the coefficients c_i are given by

$$c_i=\frac{\Gamma(i+1)}{\Gamma(i+1+\alpha)}\int_0^{\infty}x^{\alpha}e^{-x}L_i^{(\alpha)}(x)u(x)dx, \quad i=0,1,... . \tag{11}$$

Consider only the first (m + 1)-terms of generalized Laguerre polynomials, so we can write

$$u_m(x)=\sum_{i=0}^{m}c_iL_i^{(\alpha)}(x). \tag{12}$$

For more details on Laguerre polynomials, its definitions and properties see ([1], [3], [6], [16], [28], [29]).

3 The Approximate Fractional Derivatives of $L_n^{(\alpha)}(x)$ and Its Convergence Analysis

The main goal of this section is to introduce the following theorems to derive an approximate formula of the fractional derivatives of the generalized Laguerre polynomials and study the truncating error and its convergence analysis.

The main approximate formula of the fractional derivative of $u(x)$ is given in the following theorem.

Theorem 1. [13]

Let $u(x)$ be approximated by the generalized Laguerre polynomials as (12) and also suppose $\nu>0$ then, its approximated fractional derivative can be written in the following form

$$D^{\nu}(u_m(x))\cong\sum_{i=\lceil \nu \rceil}^{m}\sum_{k=\lceil \nu \rceil}^{i}c_iw_{i,k}^{(\nu)}x^{k-\nu}, \tag{13}$$

where $w_{i,k}^{(\nu)}$ is given by $w_{i,k}^{(\nu)}=\dfrac{(-1)^k}{\Gamma(k+1-\nu)}\binom{i+\alpha}{i-k}.$

Theorem 2. [13]

The Caputo fractional derivative of order ν for the generalized Laguerre polynomials can be expressed in terms of the generalized Laguerre polynomials themselves in the following form

$$D^{\nu}L_i^{(\alpha)}(x)\cong\sum_{k=\lceil \nu \rceil}^{i}\sum_{j=0}^{k-\lceil \nu \rceil}\Omega_{i,j,k}L_j^{(\alpha)}(x), \quad i=\lceil \nu \rceil,\lceil \nu \rceil+1,...,m, \tag{14}$$

where $\Omega_{i,j,k}=\displaystyle\sum_{r=0}^{j}\frac{(-1)^{r+k}(\alpha+i)!(j)!(k+\alpha-\nu+r)!}{(k-\nu)!(i-k)!(\alpha+k)!(r)!(j-r)!(\alpha+r)!}.$

Theorem 3. [13]

The error in approximating $D^{\nu}u(x)$ by $D^{\nu}u_m(x)$ is bounded by

$$\left|E_T(m)\right|\leq\sum_{i=m+1}^{\infty}c_i\Pi_{\nu}(i,j)\frac{(\alpha+1)_j}{j!}e^{x/2}, \quad \alpha\geq 0,\ x\geq 0,\ j=0,1,..., \tag{15}$$

$$\left|E_T(m)\right| \le \sum_{i=m+1}^{\infty} c_i \Pi_\nu(i,j)\left(2-\frac{(\alpha+1)j}{j!}\right)e^{x/2}, \tag{16}$$
$$-1<\alpha\le 0,\quad x\ge 0,\quad j=0,1,\ldots,$$

where

$$\Pi_\nu(i,j)=\sum_{k=\lceil\nu\rceil}^{i}\sum_{j=0}^{k-\lceil\nu\rceil}\Omega_{i,j,k},\ \&\left|E_T(m)\right|=\left|D^\nu u(x)-D^\nu u_m(x)\right|.$$

4 Procedure Solution of the Fractional Cable Equation

Consider the fractional Cable equation of type given in Eq.(1). In order to use Laguerre collocation method, we first approximate $u(x,t)$ as

$$u_m(x,t)=\sum_{i=0}^{m}u_i(t)L_i^{(\alpha)}(x). \tag{17}$$

From Eqs.(1), (17) and theorem 1 we have

$$\sum_{i=0}^{m}\frac{du_i(t)}{dt}L_i^{(\alpha)}(x)=\sum_{i=\lceil\beta\rceil}^{m}\sum_{k=\lceil\beta\rceil}^{i}u_i(t)w_{i,k}^{(\beta)}x^{k-\beta}-\mu\sum_{i=\lceil\gamma\rceil}^{m}\sum_{k=\lceil\gamma\rceil}^{i}u_i(t)w_{i,k}^{(\gamma)}x^{k-\gamma}+f(x,t). \tag{18}$$

We now collocate Eq.(18) at $(m+1-\lceil\nu\rceil)$ points x_p, $p=0,1,\ldots,m-\lceil\nu\rceil$ as

$$\sum_{i=0}^{m}\dot{u}_i(t)L_i^{(\alpha)}(x_p)=\sum_{i=\lceil\beta\rceil}^{m}\sum_{k=\lceil\beta\rceil}^{i}u_i(t)w_{i,k}^{(\beta)}x_p^{k-\beta}-\mu\sum_{i=\lceil\gamma\rceil}^{m}\sum_{k=\lceil\gamma\rceil}^{i}u_i(t)w_{i,k}^{(\gamma)}x_p^{k-\gamma}+f(x_p,t). \tag{19}$$

For suitable collocation points we use roots of the generalized Laguerre polynomial $L_{m+1-\lceil\nu\rceil}^{(\alpha)}(x)$.

Also, by substituting Eqs.(17) in the boundary conditions (2) and using the property $L_i^{(\alpha)}(0)=\binom{\alpha+i}{i}$ we can obtain $\lceil\nu\rceil$ equations

$$\sum_{i=0}^{m}u_i(t)\binom{\alpha+i}{i}=0, \tag{20}$$

$$\sum_{i=0}^{m}u_i(t)L_i^{(\alpha)}(1)=0. \tag{21}$$

Eq.(19), together with $\lceil\nu\rceil$ equations of the boundary conditions (20)-(21), give $(m+1)$ of ordinary differential equations which can be solved, for the unknowns u_i, $i=0,1,\ldots,m$; using the finite difference method, as described in the following section.

5 Numerical Results

In this section, we present a numerical example to illustrate the efficiency and the validation of the proposed numerical method when applied to solve numerically the fractional Cable equation.

Consider the FCE (1) with $\mu=1$ and the following source term

$$f(x,t)=e^{-2t}\left(-2(x^\beta-x^\gamma)-\beta!+\frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)}x^{\beta-\gamma}-\gamma!\right), \tag{22}$$

and the boundary conditions $u(0,t)=u(1,t)=0$ and the initial condition $u(x,0)=(x^\beta-x^\gamma)$. The exact solution of Eq.(1) in this case is $u(x,t)=e^{-2t}(x^\beta-x^\gamma)$.

We apply the proposed method with m=3, and approximate the solution as follows

$$u_3(x,t)=\sum_{i=0}^{3}u_i(t)L_i^{(\alpha)}(x). \tag{23}$$

Using Eq.(19) we have

$$\sum_{i=0}^{3}\dot{u}_i(t)L_i^{(\alpha)}(x_p)=\sum_{i=2}^{3}\sum_{k=2}^{i}u_i(t)w_{i,k}^{(\beta)}x_p^{k-\beta}-\sum_{i=1}^{m}\sum_{k=1}^{i}u_i(t)w_{i,k}^{(\gamma)}x_p^{k-\gamma}+f(x_p,t),\quad p=0,1, \tag{24}$$

where x_p are roots of generalized Laguerre polynomial $L_2^{(\alpha)}(x)$, i.e., $x_0=0.427124$, $x_1=3.07288$.

By using Eqs.(20)-(21) and (24) we can obtain the following system of ODEs

$$\dot{u}_0(t)+k_1\dot{u}_1(t)+k_2\dot{u}_3(t)=R_1u_1(t)+R_2u_2(t)+R_3u_3(t)+f_0(t), \tag{25}$$

$$\dot{u}_0(t)+\ell_1\dot{u}_1(t)+\ell_2\dot{u}_3(t)=Q_1u_1(t)+Q_2u_2(t)+Q_3u_3(t)+f_1(t), \tag{26}$$

$$r_0u_0(t)+r_1u_1(t)+r_2u_2(t)+r_3u_3(t)=0, \tag{27}$$

$$s_0u_0(t)+s_1u_1(t)+s_2u_2(t)+s_3u_3(t)=0, \tag{28}$$

where

$$k_1=L_1^{(\alpha)}(x_0),\quad k_2=L_3^{(\alpha)}(x_0),\quad \ell_1=L_1^{(\alpha)}(x_1),\quad \ell_2=L_3^{(\alpha)}(x_1)$$

$$R_1=-w_{1,1}^{(\beta)}x_0^{1-\beta},\quad R_2=w_{2,2}^{(\beta)}x_0^{2-\beta}-(w_{2,1}^{(\beta)}x_0^{1-\beta}+w_{2,2}^{(\alpha)}x_0^{2-\beta}),$$
$$R_3=x_0^{2-\beta}(w_{3,2}^{(\beta)}+w_{3,3}^{(\beta)}x_0)-x_0^{1-\gamma}(w_{3,1}^{(\gamma)}+w_{3,2}^{(\gamma)}x_0+w_{3,3}^{(\gamma)}x_0^2),$$
$$Q_1=-w_{1,1}^{(\beta)}x_1^{1-\beta},\quad Q_2=w_{2,2}^{(\beta)}x_1^{2-\beta}-(w_{2,1}^{(\beta)}x_1^{1-\beta}+w_{2,2}^{(\alpha)}x_1^{2-\beta}),$$
$$Q_3=x_1^{2-\beta}(w_{3,2}^{(\beta)}+w_{3,3}^{(\beta)}x_1)-x_1^{1-\gamma}(w_{3,1}^{(\gamma)}+w_{3,2}^{(\gamma)}x_1+w_{3,3}^{(\gamma)}x_1^2),$$
$$r_i=\binom{\alpha+i}{i},\quad s_i=L_i^{(\alpha)}(1),\quad i=0,1,2,3.$$

TABLE 1
The absolute error between the exact solution and the approximate solution at $m = 3$, $m = 5$ and $T = 2$.

x	$\left\|u_{ex} - u_{approx}\right\|$ at $m = 3$	$\left\|u_{ex} - u_{approx}\right\|$ at $m = 5$
0.0	4.483787e-03	2.726496e-05
0.1	4.479660e-03	3.455890e-05
0.2	4.201329e-03	3.809670e-05
0.3	3.695172e-03	3.809103e-05
0.4	3.007566e-03	3.514280e-05
0.5	2.184889e-03	3.009263e-05
0.6	1.273510e-03	2.387121e-05
0.7	0.319831e-03	1.735125e-05
0.8	0.629793e-03	1.119821e-05
0.9	1.528978e-03	0.572150e-05
1.0	2.331347e-03	0.072566e-05

Now, to use FDM for solving the system (25)-(28), we will use the following notations $t_i = i\tau$ to be the integration time $0 \le t_i \le T$, $\tau = T/N$ for $i = 0,1,\dots,N$. Define $u_i^n = u_i(t_n)$, $f_i^n = f_i(t_n)$. Then the system (25)-(28), is discretized in time and takes the following form

$$\frac{u_0^{n+1} - u_0^n}{\tau} + k_1 \frac{u_1^{n+1} - u_1^n}{\tau} + k_2 \frac{u_3^{n+1} - u_3^n}{\tau} = R_1 u_1^{n+1} + R_2 u_2^{n+1} + R_3 u_3^{n+1} + f_0^{n+1}, \quad (29)$$

$$\frac{u_0^{n+1} - u_0^n}{\tau} + \ell_1 \frac{u_1^{n+1} - u_1^n}{\tau} + \ell_2 \frac{u_3^{n+1} - u_3^n}{\tau} = Q_1 u_1^{n+1} + Q_2 u_2^{n+1} + Q_3 u_3^{n+1} + f_1^{n+1}, \quad (30)$$

$$r_0 u_0^{n+1} + r_1 u_1^{n+1} + r_2 u_2^{n+1} + r_3 u_3^{n+1} = 0, \quad (31)$$

$$s_0 u_0^{n+1} + s_1 u_1^{n+1} + s_2 u_2^{n+1} + s_3 u_3^{n+1} = 0. \quad (32)$$

We can write the above system (29)-(32) in the following matrix form as follows

$$\begin{pmatrix} 1 & k_1 - \tau R_1 & -\tau R_2 & k_2 - \tau R_3 \\ 1 & \ell_1 - \tau Q_1 & -\tau Q_2 & \ell_2 - \tau Q_3 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & k_1 & 0 & k_2 \\ 1 & \ell_1 & 0 & \ell_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}^{n} + \tau \begin{pmatrix} f_0 \\ f_1 \\ 0 \\ 0 \end{pmatrix}^{n+1}. \quad (33)$$

We use the notation for the above system

$$AU^{n+1} = BU^n + F^{n+1}, \text{ or } U^{n+1} = A^{-1}BU^n + A^{-1}F^{n+1}, \quad (34)$$

where $U^n = (u_0^n, u_1^n, u_2^n, u_3^n)^T$ and $F^n = (\tau f_0^n, \tau f_1^n, 0, 0)^T$.

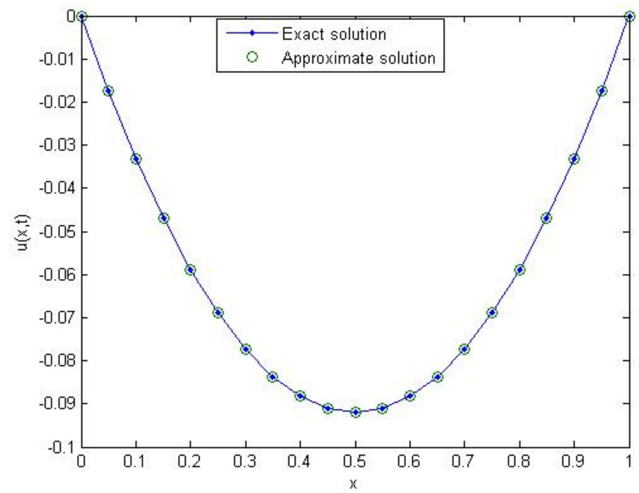

Figure 1. Comparison between the exact solution and the approximate solution at $T = 0.5$ with $\tau = 0.0025$, $m = 3$.

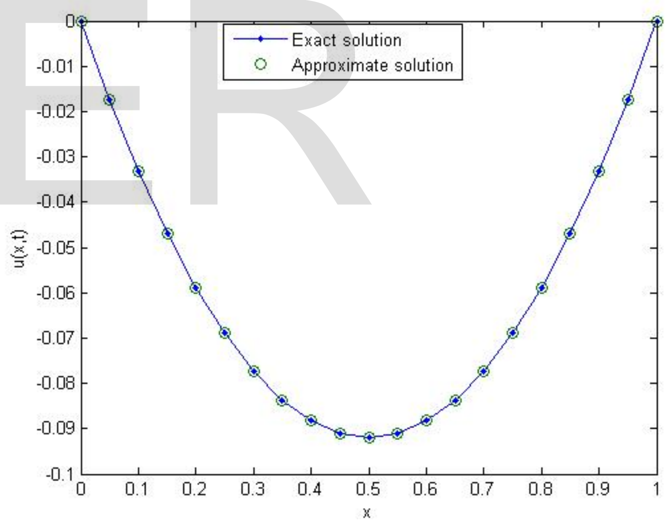

Figure 2. Comparison between the exact solution and the approximate solution at $T = 0.5$ with $\tau = 0.0025$, $m = 5$.

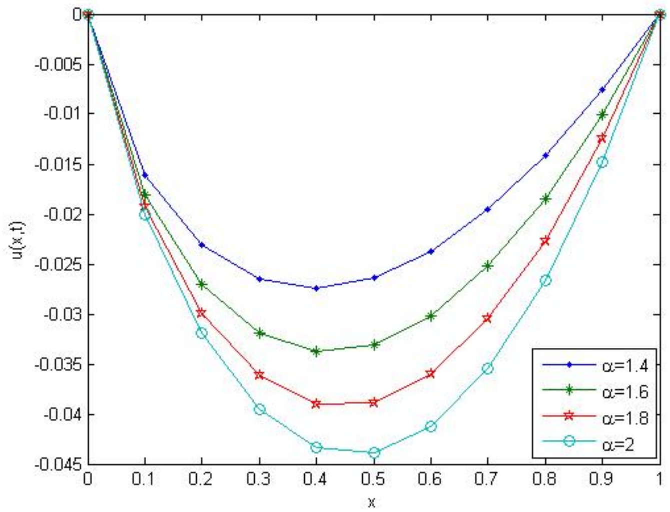

Figure 3. The behavior of the approximate solution at different values of β at $\gamma = 0.8$.

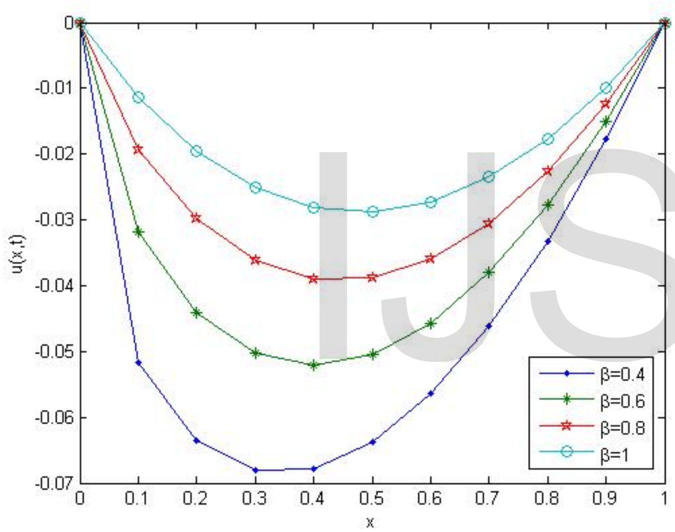

Figure 4. The behavior of the approximate solution at different values of γ at $\beta = 1.8$.

The obtained numerical results by means of the proposed method are shown in table 1 and figures 1-4. In table 1, the absolute error between the exact solution u_{ex} and the approximate

solution u_{approx} at $m = 3$ and $m = 5$ with the final time $T = 2$ are given. Also, in figures 1 and 2, comparison between the exact solution and the approximate solution at $T = 0.5$ with time step $\tau = 0.0025$, $m = 3$ and $m = 5$ are presented, respectively. Also, in figures 3 and 4, the behavior of the approximate solution at $T = 0.5$ and $m = 5$ with different values of β and γ are presented, respectively. From, these figures, we can see that the behavior of the approximate solution depends on the order of the fractional derivative.

6 Conclusion

The properties of the Laguerre polynomials are used to reduce the fractional Cable equation to the solution of system of ODEs which solved by using FDM. The fractional derivative is considered in the Caputo sense. In this article, special attention is give to study the convergence analysis and estimate an upper bound of the error for the proposed approximate formula of the fractional derivative. The solution obtained using the suggested method is in excellent agreement with the already existing ones and show that this approach can be solved the problem effectively. From the resulted numerical solution, we can conclude that the used techniques in this work are appli to solve many other problems. It is evident that the overall errors can be made smaller by adding new terms from the series (23). Comparisons are made between the approximate solution and the exact solution to illustrate the validity and the great potential of the technique. All computations in this paper are done using Matlab 8.

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References

[1] R. Askey and G. Gasper, "Convolution structures for Laguerre polynomials", *J. d'Anal. Mathematics*, 31, (48-68), 1977.
[2] A. V. Chechkin, V. Y. Gonchar, J. Klafter, R. Metzler and L. V. Tanatarov, "Le'vy flights in a steep potential well", *J. Stat. Phys.*, 115, (1505-1535), 2004.
[3] X. Cheng-Long and G. Ben-Yu, "Laguerre pseudo-spectral method for non-linear partial differential equations", *J. Computational Mathematics*, 20, (413-428), 2002.
[4] B. I. Henry, T. A. M. Langlands, and S. L. Wearne, "Fractional Cable models for spiny neuronal dendrites", *Physical Review Letters*, 100(12), (2008), Article ID 128103, 4.
[5] H. Jafari and V. Daftardar-Gejji, "Solving linear and nonlinear fractional diffusion and wave equations by ADM", *Applied Mathematics and Computation*, 180, (488-497), 2006.
[6] I. Z. Khabibrakhmanov and D. Summers, "The use of generalized Laguerre polynomials in spectral methods for non-linear differential equations, *Computers Mathematics Application*, 36, (65-70), 1998.
[7] M. M. Khader, "On the numerical solutions for the fractional diffusion equation", *Communications in Nonlinear Science and Numerical Simulation*, 16, (2535-2542), 2011.
[8] M. M. Khader, "Introducing an efficient modification of the VIM by using Chebyshev polynomials", *Application and Applied Mathematics: An International Journal*, 7(1), (283-299), 2012.
[9] M. M. Khader, "Introducing an efficient modification of the HPM by using Chebyshev polynomials", *Arab Journal of Mathematical Sciences*, 18, (61-71), 2012.
[10] M. M. Khader and A. S. Hendy, "The approximate and exact solutions of the fractional-order delay differential equations using Legendre pseudospectral method", *International Journal of Pure and Applied Mathematics*, 74(3), (287-297), 2012.
[11] M. M. Khader and A. S. Hendy, "A numerical technique for solving

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fractional variational problems“, *Mathematical Methods in Applied Sciences*, 36(10), (1281-1289), 2013.

[12] M. M. Khader, T. S. EL Danaf and A. S. Hendy, “A computational matrix method for solving systems of high order fractional differential equations“, *Applied Mathematical Modelling*, 37, (4035-4050), 2013.

[13] M. M. Khader, T. S. EL Danaf and A. S. Hendy, “Efficient spectral collocation method for solving multi-term fractional differential equations based on the generalized Laguerre polynomials“, *J. of Fractional Calculus and Applications*, 3(13), (1-14), 2012.

[14] M. M. Khader, “An efficient approximate method for solving linear fractional Klein-Gordon equation based on the generalized La guerre polynomials“, *International Journal of Computer Mathemat ics*, 90(9), (1853-1864), 2013.

[15] T. A. M. Langlands, B. I. Henry and S. L.Wearne, “Fractional Cable equation models for anomalous electro-diffusion in nerve cells: infinite domain solutions“, *Journal of Mathematical Biology*, 59(6), (761-808), 2009.

[16] Z. Lewandowski and J. Szynal, An upper bound for the Laguerre polynomials, *Journal of Computational and Applied Mathematics*, 99, (529-533), 1998.

[17] I. Podlubny, “*Fractional Differential Equations*“, Academic Press, New York, 1999.

[18] J. Quintana-Murillo and S. B. Yuste, “An explicit numerical method for the fractional Cable equation“, *Int. J. of Differential Equations*, 2011, Article ID 231920, 12 pages, 2011

[19] W. Rall, “Core conductor theory and Cable properties of neurons, in: R. Poeter (Ed.), Handbook of Physiology: The Nervous System“, Vol. 1, American Physiological Society, Bethesda, Md, 39-97, chapter 3, 1977.

[20] G. D. Smith, “*Numerical Solution of Partial Differential Equations*“, Oxford University Press, 1965.

[21] I. M. Sokolov, J. Klafter and A. Blumen, “Fractional kinetics“, *Phys. Today*, 55, (48-54), 2002.

[22] N. H. Sweilam and M. M. Khader, A Chebyshev pseudo-spectral method for solving fractional integro-differential equations, *ANZIAM*, 51, (464-475), 2010.

[23] N. H. Sweilam, M. M. Khader and A. M. Nagy, “Numerical solution of two-sided space-fractional wave equation using finite difference method“, *Journal of Computional and Applied Mathematics*, 235, (2832-2841), 2011.

[24] N. H. Sweilam, M. M. Khader and M. Adel, “On the stability analysis of weighted average finite difference methods for frac tional wave equation“, *Fractional Differential Calculus*, 2(1), (17-29), 2012.

[25] N. H. Sweilam, M. M. Khader and A. M. S. Mahdy, “Crank-Nicolson finite difference method for solving time-fractional diff usion equation“, *Journal of Fractional Calculus and Applications*, 2(2), (1-9), 2012.

[26] N. H. Sweilam, M. M. Khader and A. M. S. Mahdy, “Numerical stdies for solving fractional-order Logistic equation, *Int. J. of Pure and Applied Mathematics*, 78(8), (1199-1210), 2012.

[27] N. H. Sweilam, M. M. Khader and A. M. S. Mahdy, Numerical stu dies for fractional-order Logistic differential equation with two different delays“, *Journal of Applied Mathematics*, 2012, Article ID 764894, 14 pages.

[28] F. Talay Akyildiz, Laguerre spectral approximation of Stokes first problem for third-grade fluid“, *J. Computational Mathematics*, 10, (1029-1041), 2009.

[29] L. Wang and Benyu Guo, Stair Laguerre pseudospectral method for differential equations on the half line, *Advanced Computational Mathematics*, 25, (305-322), 2006.